$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}} - \sqrt{30 + 12\sqrt{5}} - \sqrt{30 - 12\sqrt{5}})$$

$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}(1 - \sqrt{3})(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}}).$$

By simplifying the expression in the last parenthesis and then collecting under the radical sign, we have, after a simple reduction,

$$\sin 3^\circ = \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}}).$$

Also solved by L. E. Mensenkamp, J. L. Riley, Horace Olson, H. S. Uhler, H. C. Feemster, Florence Rae, R. M. Mathews, H. L. Agard, Gilbert A. Ault, Frank Irwin, C. E. Githens, A. M. Harding, and the Proposer.

491 (Algebra). Proposed by J. W. LASLEY, University of North Carolina.

Solve the equations $xy = x^2 - y^2$ and $x^2 + y^2 = x^3 - y^3$ for x and y.

Solution by J. W. Baldwin, Ann Arbor, Michigan.

Solving $xy = x^2 - y^2$ for x in terms of y we have $x = \frac{1}{2}(1 + \sqrt{5})y$ and $x = \frac{1}{2}(1 - \sqrt{5})y$. These values substituted in $x^2 + y^2 = x^3 - y^3$ give, after simplification, $y^2(y - \frac{1}{2}\sqrt{5}) = 0$ (1) and $y^2(y + \frac{1}{2}\sqrt{5}) = 0$ (2). From (1), $y = 0, 0, \frac{1}{2}\sqrt{5}$ and from (2) $y = 0, 0, -\frac{1}{2}\sqrt{5}$. Hence, the corresponding values of x are $0, 0, (5 + \sqrt{5})/4$ and $0, 0, (5 - \sqrt{5})/4$. From (1) or (2) it is seen that two branches of the curve represented by the second equation pass through the origin. It is readily determined that these branches are imaginary and, hence, the origin is a conjugate point.

Also solved by S. E. Rasor, F. H. Holestin, Elgin E. Groseclose, H. N. Carleton, A. M. Harding, J. L. Riley, Polycarp Hansen, E. B. Escott, G. Y. Sosnow, J. Q. McNatt, O. S. Adams, Horace Olson, T. C. Amick, L. E. Lunn, Pauline Sperry and the Proposer.

433 (Calculus). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} - \frac{y}{x} = 0.$$

I. Solution by Emil L. Post, New York City.

We have

$$x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = y. \tag{1}$$

Operating on both sides by $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ (see next problem), we have

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}. \tag{2}$$

But for any operation $f(D)$

$$f(D)u\cdot v = uf(D)v + \frac{du}{dx}\frac{f'(D)v}{1!} + \frac{d^2u}{dx^2}\frac{f''(D)v}{2!} + \cdots.$$

Let

$$f(D) = D^{\frac{1}{2}} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}; \qquad u = x; \qquad v = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}.$$

Then

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + \frac{1}{2}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = x\frac{dy}{dx} + \tfrac{1}{2}y. \tag{3}$$

Substituting from the original equation and (3) into (2), we have

$$x\frac{dy}{dx} + \tfrac{1}{2}y = \frac{y}{x}. \tag{4}$$

From this we have

$$\log y = -1/x - \tfrac{1}{2}\log x + \log C,$$

or

$$y = Ce^{-(1/x)}x^{-\frac{1}{2}}.$$

II. Solution by the Proposer.

According to Professor Kelland (*Trans. Royal Society of Edinburgh*, Vols. XIV and XVI) the general differential operator may be defined as follows:

$$\frac{d^{\mu}x^{n}}{dx^{\mu}} = (-1)^{\mu}\frac{\Gamma(-n+\mu)}{\Gamma(-n)}x^{n-\mu},$$

for all values of n and μ. We have $\Gamma(n+1) = n\Gamma(n)$.

Let us assume $y = A_0 + A_1x^{-\frac{1}{2}} + A_2x^{-1} + A_3x^{-\frac{3}{2}} + \cdots$.

Then

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\left\{\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1x^{-1} + \frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2x^{-\frac{3}{2}} + \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3x^{-2} + \cdots\right\}$$

and

$$\frac{y}{x} = A_0x^{-1} + A_1x^{-\frac{3}{2}} + A_2x^{-2} + A_3x^{-\frac{5}{2}} + \cdots.$$

Hence,

$$i\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1 = A_0 \text{ and } A_1 = \frac{\sqrt{\pi}}{i}A_0 = -i\sqrt{\pi}A_0.$$

For, since $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$ when p is a fraction less than one, then $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Also, $\Gamma(1) = 1$.

Furthermore,

$$i\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2 = A_1 = -i\sqrt{\pi}A_0, \text{ or } A_2 = -2A_0,$$

since $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$; also

$$i\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3 = A_2 = -2A_0, \text{ or } A_3 = -\frac{\sqrt{\pi}}{i}A_0 = i\sqrt{\pi}A_0.$$

Finally, $y = A_0(1 - i\sqrt{\pi}x^{-\frac{1}{2}} - 2x^{-1} + i\sqrt{\pi}x^{-\frac{3}{2}} + \cdots)$.

Note.—This problem, which is very similar to one solved by Professor Kelland, was submitted because it was thought that it might prove of interest to the readers of the Monthly. The proposer would be glad to see some discussion of the subject of general differentiation.

The above solution may be considered as a reply to the following question:

360 (Calculus.) Proposed by ELMER SCHUYLER, Brooklyn, New York.

What interpretation must be given to

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} \text{ so that } \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right) = \frac{dy}{dx}?$$

435 (Calculus). Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^{\infty} e^{-x^2-a^2/x^2}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation, rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, page 106–107.